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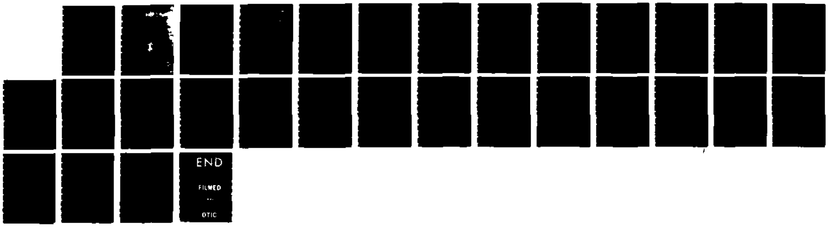

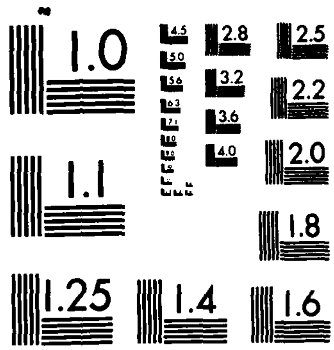

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FORCE-FREE MAGNETIC FIELDS

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Report No. CSSA-ASTRO-85-25

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CENTER FOR SPACE SCIENCE AND ASTROPHYSICS
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Abstract

The problem under discussion is that of calculating magnetic field configurations in which the Lorentz force $\vec{j} \times \vec{B}$ is everywhere zero, subject to specified boundary conditions. We choose to represent the magnetic field in terms of Clebsch variables in the form $\vec{B} = \nabla\alpha \times \nabla\beta$. These variables are constant on any field line so that each field line is labeled by the corresponding values of α and β. When the field is described in this way, the most appropriate choice of boundary conditions is to specify the values of α and β on the bounding surface. We show that such field configurations may be calculated by a "magneto-frictional" method. We imagine that the field lines move through a stationary medium, and that each element of magnetic field is subject to a frictional force parallel to and opposing the velocity of the field line. This concept leads to an iteration procedure for modifying the variables α and β, that tends asymptotically towards the force-free state. We apply the method first to a simple problem in two rectangular dimensions, and then to a problem of cylindrical symmetry that was previously discussed by Barnes and Sturrock (1972). In one important respect, our new results differ from the earlier results of Barnes and Sturrock, and we conclude that the earlier article was in error.

I. INTRODUCTION

Magnetic fields play a key role in all solar activity and probably also in similar phenomena on other stars and in other astrophysical objects. Knowledge of the quiescent structure of the magnetic field is obviously an indispensable prerequisite for any study of the dynamics of these fields. In general, the equilibrium configuration involves a balance of several forces including the Lorentz force, the plasma-pressure gradient, and the gravitational force acting on the plasma. However, in many cases we are dealing with a "low-beta" situation in which the gas pressure is small compared with the magnetic pressure. Since the distribution of gas along a magnetic field line is determined by hydrostatic considerations, the gravitational potential energy density will be comparable with the plasma pressure. Under these conditions the magnetic-field configuration will approximate a force-free state for which the Lorentz force $\vec{F}$ is zero, i.e.,

$$\vec{F} \equiv \vec{j} \times \vec{B} = 0. \tag{1.1}$$

It is clear that the general problem represented by equation (1.1) is nonlinear, since the sum of

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two separate solutions of equation (1.1) does not, in general, produce a third solution. However, a special class of force-free fields may be generated from solutions of the linear equation

$$\nabla \times \vec{B} = \lambda \vec{B}, \tag{1.2}$$

where λ is taken to be a constant (Priest 1982). In this case, it is clear that the sum of two solutions of equation (1.2) generates a third solution. As a result, one can develop a set of solutions of equation (1.1) in terms of convenient functions, and then manipulate a combination of these solutions to match specified boundary conditions.

Suppose, however, that we are trying to find the force-free field configuration produced by a finite source such as a pair of sunspots of opposite polarity. If we investigate the field at a large distance from the source, we expect that the gradients will vary as Br^{-1}. However, such a variation of $\vec{j}$ with respect to r is clearly incompatible with the assumption that λ is constant. Hence the assumption that $\lambda = constant$, which reduces the nonlinear problem of (1.1) to the linear problem described by (1.2), is inappropriate for the investigation of a wide range of interesting problems of solar physics and astrophysics.

In the special case that the magnetic-field configuration is independent of one of the spatial coordinates, it is possible to generate solutions of the nonlinear equation (1.1) by a special technique described by Priest (1982). For instance, for field configurations that are independent of z in rectangular cartesian coordinates, it is possible to generate a set of nonlinear force-free fields by specifying that the field is of the form

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}, \quad B_z = F(A_z), \tag{1.3}$$

where $F(A_z)$ is a specified function of A , and the vector potential satisfies the equation

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + F(A_z)F'(A_z) = 0. \tag{1.4}$$

However, this is simply a technique for generating a special set of solutions of the nonlinear equation, and is therefore inappropriate for solving a general problem, such as computing the force-free field that satisfies certain boundary conditions.

Force-free fields arise in situations in which the electrical conductivity is sufficiently high that the relative diffusion of plasma and magnetic field may be neglected. In this situation, Alfvén's concept of "frozen-in magnetic field" is appropriate (Cowling 1957), and it is convenient to think of magnetic field lines as having physical reality. In view of the equation

$$\nabla \cdot \vec{B} = 0, \tag{1.5}$$

it is possible to describe the magnetic field in terms of the Clebsch variables

$$\vec{B} = \nabla\alpha \times \nabla\beta, \tag{1.6}$$

from which it is obvious that

$$\vec{B} \cdot \nabla\alpha = 0, \quad \vec{B} \cdot \nabla\beta = 0, \tag{1.7}$$

so that α and β may be used to "label" the field lines of the configuration (Parker 1979).

It is worth noting that one could employ, instead of equation (1.6), the equation

$$\vec{B} = f(\alpha, \beta)\, \nabla\alpha \times \nabla\beta, \tag{1.8}$$

which appears to be more general but in fact corresponds to an implicit transformation of the variables α, β to another pair of such variables.

In terms of the description (1.6), one finds that the nonlinear vector equation (1.1) reduces to the pair of scalar nonlinear equations

$$\vec{j} \cdot \nabla\alpha = 0, \quad \vec{j} \cdot \nabla\beta = 0, \tag{1.9}$$

which are clearly consistent with equations (1.7) for a force-free field, for which $\vec{j}$ is parallel to $\vec{B}$.

In terms of α and β, equations (1.9) become

$$\begin{aligned} [\nabla \times (\nabla\alpha \times \nabla\beta)] \cdot \nabla\alpha &= 0, \\ [\nabla \times (\nabla\alpha \times \nabla\beta)] \cdot \nabla\beta &= 0, \end{aligned} \tag{1.10}$$

which look rather formidable when written out explicitly. Nevertheless, these equations may be solved by a relaxation procedure, which was the approach adopted by Sturrock and Woodbury (1967) for a problem in rectangular Cartesian coordinates and by Barnes and Sturrock (1972) for a problem in cylindrical coordinates. In the simplest version of the relaxation procedure, one attempts to adjust α and β at each mesh point in turn to arrange that equations (1.10) are satisfied at that mesh point. An alternative procedure is to attempt to relax simultaneously an array of points, such as those on a line, but this requires inversion of a matrix. It was by no means clear what was the best numerical procedure for ensuring both numerical stability and the most rapid convergence.

Sakurai (1979) has proposed a different procedure aimed at the calculation of general three-dimensional (3D) force-free magnetic-field configurations. Sakurai also describes the field in terms of Clebsch variables, but he uses the fact that the force-free field satisfies the variation principle

$$\delta\Big[\frac{1}{8\pi} \int_V d^3x\, (\nabla\alpha \times \nabla\beta)^2\Big] = 0. \tag{1.11}$$

Sakurai takes advantage of the variation principle to employ the Rayleigh-Ritz method (Courant and Hilbert 1953) of representing the field in terms of a set of base functions and solving for the coefficients arising in this representation.

In another article, Sakurai (1981) has discussed another method for computing force-free fields. If B_n, the normal value of the magnetic field is given on a bounding surface, and if λ is specified

either in the region of the surface where B_n is positive or in the region where B_n is negative, the magnetic field can be found by an iterative process. Starting from the potential field with the given B_n, the value of λ is used to distribute currents along the field lines. The field is then recalculated, and the procedure repeated until the calculation converges.

II. THE MAGNETO-FRICTIONAL METHOD

We have recently begun to explore what first appeared to be an entirely different approach to this problem. This technique, we find, is closely related to a method developed by Chodura and Schlüter (1981) for application to MHD problems.

Suppose that there is a frictional force acting on the plasma if it moves with a non-zero velocity with respect to a fictitious fixed medium. Then the equation of motion of the plasma is

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) = -\nabla p + \rho \vec{g} + \vec{F} - \nu \vec{v}, \tag{2.1}$$

where $\vec{F}$ again represents the Lorentz force, ρ and p are density and pressure, $\vec{g}$ is the gravitational field, and ν is the coefficient of friction. In the situations that lead to force-free fields, ρ and p are negligible, so that equation (2.1) leads to

$$\vec{v} = \nu^{-1} \vec{F}. \tag{2.2}$$

We are considering the case that the plasma has high electrical conductivity so that

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}). \tag{2.3}$$

This equation is satisfied if we require that α and β "move with the plasma," i.e., that

$$\begin{aligned} \frac{\partial \alpha}{\partial t} + \vec{v} \cdot \nabla \alpha = 0, \\ \frac{\partial \beta}{\partial t} + \vec{v} \cdot \nabla \beta = 0. \end{aligned} \tag{2.4}$$

On combining equations (2.2) and (2.4), we obtain the equations

$$\begin{aligned} \frac{\partial \alpha}{\partial t} = -\nu^{-1} \vec{F} \cdot \nabla \alpha, \\ \frac{\partial \beta}{\partial t} = -\nu^{-1} \vec{F} \cdot \nabla \beta, \end{aligned} \tag{2.5}$$

where ν is so far an arbitrary function of space and time. In practice, it has proved convenient to choose the expression ν so that equations (2.5) lead to the following increments for α and β for each time step:

$$\begin{aligned} \delta\alpha = -\mu \frac{\vec{F} \cdot \nabla \alpha}{B^2}, \\ \delta\beta = -\mu \frac{\vec{F} \cdot \nabla \beta}{B^2}, \end{aligned} \tag{2.6}$$

where μ is an adjustable parameter.

The techniques represented by equations (1.10) and (2.6) for computing nonlinear force-free fields appear to be quite different, yet this proves not to be the case. If we consider, for simplicity, a rectangular coordinate system with mesh size h, Woodbury (1973) showed that the variations $\delta\alpha, \delta\beta$ require to satisfy equations (1.10) are given by

$$\begin{pmatrix} |\nabla\beta|^2 & -\nabla\alpha\cdot\nabla\beta \\ \nabla\alpha\cdot\nabla\beta & |\nabla\alpha|^2 \end{pmatrix} \begin{pmatrix} \delta\alpha \\ \delta\beta \end{pmatrix} = -\frac{h^2}{6} \begin{pmatrix} \vec{j}\cdot\nabla\beta \\ -\vec{j}\cdot\nabla\alpha \end{pmatrix}. \tag{2.7}$$

We find that the determinant of the matrix is given by

$$\Delta = |\nabla\alpha|^2|\nabla\beta|^2 - (\nabla\alpha\cdot\nabla\beta)^2 = B^2, \tag{2.8}$$

so that equation (2.7) leads to

$$\begin{pmatrix} \delta\alpha \\ \delta\beta \end{pmatrix} = -\frac{h^2}{6B^2} \begin{pmatrix} |\nabla\alpha|^2 & \nabla\alpha\cdot\nabla\beta \\ -\nabla\alpha\cdot\nabla\beta & |\nabla\beta|^2 \end{pmatrix} \begin{pmatrix} \vec{j}\cdot\nabla\beta \\ -\vec{j}\cdot\nabla\alpha \end{pmatrix}. \tag{2.9}$$

This equation may be rewritten as

$$\begin{pmatrix} \delta\alpha \\ \delta\beta \end{pmatrix} = -\frac{h^2}{6B^2} \begin{pmatrix} \vec{F}\cdot\nabla\alpha \\ \vec{F}\cdot\nabla\beta \end{pmatrix}, \tag{2.10}$$

which is clearly of the same form as equation (2.6).

In practice, it is found advantageous to "under-relax" by introducing into the coefficient of (2.10) an additional multiplying constant less than unity. Hence, although the relaxation approach and the magneto-frictional approach begin from quite different premises, they lead eventually to similar equations for computing force-free fields.

III. ENERGY THEOREM

According to our model, the only form of stored energy is magnetic, and energy is continually decreasing since the magnetic field lines are continually doing work in moving against friction. If W is the total magnetic energy,

$$W = \frac{1}{8\pi}\int_V d^3x\, B^2, \tag{3.1}$$

we see that

$$\frac{dW}{dt} = -\int_V d^3x\, \vec{v}\cdot\vec{F}. \tag{3.2}$$

On using equation (2.2), this becomes

$$\frac{dW}{dt} = -\int_V d^3x\, \nu^{-1} F^2. \tag{3.3}$$

This confirms our conjecture that the magnetic energy will decrease monotonically, and that $dW/dt = 0$ only when $\vec{F} = 0$ everywhere, that is, the magnetic field is force-free.

Suppose that the force-free magnetic field, for given boundary conditions restricting α and β, is $\vec{B}_0$. Consider a perturbation from the force-free state, the magnitude of the perturbation being determined by the small parameter ϖ. Then we may write

$$\vec{B} = \vec{B}_0 + \varpi\vec{B}_1 + \varpi^2\vec{B}_2 + \ldots \tag{3.4}$$

As a result of the variational theorem, equation (1.11),

$$\int d^3x\, \vec{B}_0 \cdot \vec{B}_1 = 0, \quad \int d^3x\, \vec{B}_0 \cdot \vec{B}_2 = 0, \ldots \tag{3.5}$$

so that, in the expansion

$$W = W_0 + \varpi W_1 + \varpi^2 W_2 + \ldots\ , \tag{3.6}$$

$$W_1 = 0 \quad and \ W_2 = \frac{1}{8\pi}\int d^3x\, B_1^2. \tag{3.7}$$

However, since $\vec{F}_0 \equiv 0$, we see from (3.3) that

$$\frac{dW_2}{dt} = -\int d^3x\, \nu^{-1} F_1^2. \tag{3.8}$$

Hence, to lowest order in the perturbation,

$$\frac{1}{\Delta W}\frac{d}{dt}\Delta W = -\frac{\int d^3x\, \nu^{-1} F_1^2}{\frac{1}{8\pi}\int d^3x\, B_1^2}, \tag{3.9}$$

where

$$W = W_0 + \Delta W. \tag{3.10}$$

It is clear from equation (3.9) that those perturbations of the force-free field that have the highest wave number will be those that damp most rapidly. Hence "local" fluctuations will damp rapidly, but "global" perturbations will damp slowly.

In order to avoid numerical instability, one must adopt smaller time steps as one chooses a smaller spatial grid size. For these reasons, it is advantageous to begin with a coarse mesh since this makes it possible to get rid of perturbations of large wave number in a reasonable number of time steps. One may then choose successively smaller grid sizes since one is then getting rid of perturbations of increasingly higher wave number. One of the advantages of the magneto-frictional viewpoint is that it suggests that one adopt a regridding procedure, and this is found to be highly advantageous.

One may specify an eigenvalue problem by looking for perturbations that, to lowest order, satisfy the equations

$$\begin{aligned} \dot{\alpha}_1 + \sigma\alpha_1 &= 0, \\ \dot{\beta}_1 + \sigma\beta_1 &= 0, \end{aligned} \tag{3.11}$$

and requiring that the perturbations α_1,β_1 are zero on the boundary S,on which α and β are specified. We see from equation (2.5) that these equations may be written explicitly as

$$-\frac{1}{4\pi\nu}[(\nabla\times(\nabla\alpha_0\times\nabla\beta_0))\times(\nabla\alpha_0\times\nabla\beta_1+\nabla\alpha_1\times\nabla\beta_0)$$
$$+(\nabla\times(\nabla\alpha_0\times\nabla\beta_1+\nabla\alpha_1\times\nabla\beta_0))\times(\nabla\alpha_0\times\nabla\beta_0)]\cdot\nabla\alpha_0+\sigma\alpha_1=0, \qquad (3.12a)$$

$$-\frac{1}{4\pi\nu}[(\nabla\times(\nabla\alpha_0\times\nabla\beta_0))\times(\nabla\alpha_0\times\nabla\beta_1+\nabla\alpha_1\times\nabla\beta_0)$$
$$+(\nabla\times(\nabla\alpha_0\times\nabla\beta_1+\nabla\alpha_1\times\nabla\beta_0))\times(\nabla\alpha_0\times\nabla\beta_0)]\cdot\nabla\beta_0+\sigma\beta_1=0. \qquad (3.12b)$$

It is clear that the eigenfunction corresponding to the smallest value of the eigenvalue σ is the one that will dominate asymptotically in determining the rate of decrease of the magnetic energy. Hence we expect that, asymptotically, the magnetic energy will decrease exponentially.

IV. TEST CASE

In order to study the variation, during the iteration process, of the departure of the calculated magnetic field from the force-free field, it was convenient to base a test case on a known force-free configuration. The following functional form for α and β

$$\begin{aligned}\alpha(x,z) &= -\frac{B_0}{k}e^{-lz}\cos kx,\\ \beta(x,y,z) &= y-\gamma(x,z) = y-(\frac{k^2}{l^2}-1)^{1/2}x\end{aligned} \qquad (4.1)$$

yields the configuration

$$\begin{aligned}B_x &= -\frac{l}{k}B_0e^{-lz}\cos kx,\\ B_y &= -(1-\frac{l^2}{k^2})^{1/2}B_0e^{-lz}\cos kx,\\ B_z &= B_0e^{-lz}\sin kx,\end{aligned} \qquad (4.2)$$

which is readily shown to be force-free. We adopted the correct values of α and γ on the boundaries of a rectangle, but we added a large perturbation in the interior so that the initial functions were in error by about a factor of two.

In Figure 1 we show (a) the initial trial form for the potential α, and (b,c,d) the form after 5, 15, and 60 iterations, respectively. Figure 1(d) is very close to the exact solution. After 100 iterations, the calculated contours are indistinguishable from the exact contours, and the Lorentz force differs from zero by only about 10^{-3} of its initial magnitude. These calculations have also been carried out using a 32×32 grid. Even in this case, the calculations take only a few seconds on a VAX/750.

Figure 2 is a plot of the variation of the excess of the estimated magnetic energy over its exact value (given by the analytical solution) as the iteration proceeds. These early iterations were carried out with a coarse (8×8) mesh. We see that the convergence is indeed close to exponential.

V. CYLINDRICAL MODEL

Barnes and Sturrock (1972) published the results of a calculation of a cylindrically symmetric nonlinear force-free field configuration. The model was chosen to resemble the field that would be developed if a sunspot were to rotate with respect to the surrounding region of opposite magnetic polarity. Since the model adopted by Barnes and Sturrock has a mild singularity at the origin, we modify it slightly to avoid that singularity. Using z, r, ϕ coordinates, we assume that $\alpha = \alpha(z, r)$ and that

$$\beta = \phi - \gamma(z, r). \tag{5.1}$$

The function γ traces the variation of ϕ along a field line since $\beta = const.$ on a field line. We find that

$$B_z = \frac{1}{r}\frac{\partial \alpha}{\partial r}, \quad B_r = -\frac{1}{r}\frac{\partial \alpha}{\partial z}, \quad B_\phi = \frac{\partial \alpha}{\partial r}\frac{\partial \gamma}{\partial z} - \frac{\partial \alpha}{\partial z}\frac{\partial \gamma}{\partial r}, \tag{5.2}$$

so that $2\pi\alpha(z, r)$ is the magnetic flux embraced by the circle $z = const., r = const.$, provided that we adopt $\alpha(z, 0) = 0$.

We have adopted the following boundary conditions for α and γ on the plane $z = 0$:

$$\begin{aligned} \alpha(0, r) &= B_0 r^2 e^{-r^2/R^2}, \\ \gamma(0, r) &= \begin{cases} 0.5\gamma_M cos(\pi r/2R), & 0 \le r \le 2R; \\ -0.5\gamma_M, & r \ge 2R. \end{cases} \end{aligned} \tag{5.3}$$

As we see from Figure 3, the magnetic polarity reverses at $r = R$, and there is a relative rotation between the negative polarity region and the positive polarity region, of maximum value γ_M.

We have calculated a sequence of magnetic-field configurations for the values $\gamma_M = 0, 0.5\pi, \pi, 1.5\pi, 2\pi, 2.5\pi$, and 3π. It was necessary to close the region of calculation by setting up supplementary boundaries at $z = L$ and $r = L$. The results of our calculations are shown in Figure 4. The appropriate choice of the outer boundary depends sensitively on the value of γ_M. The choice $L = 5R$ is quite adequate for $\gamma_M = 0$, but quite unacceptable for $\gamma_M = 0.5\pi$. The choice $L \sim 20R$, adopted in calculations leading to Figure 4, is adequate for $\gamma_M = 2\pi$, but severely distorts the outer field lines for the case $\gamma_M = 3\pi$.

We have computed the total magnetic energy for each case. The result is listed in Table I, together with our estimate of the total magnetic energy of the corresponding open-field configuration. We see that, as the stress of the force-free field increases, the total energy approaches but does not exceed that of the corresponding open-field configuration. There is an important difference between these results of the present article and those of an earlier article by Barnes and Sturrock (1972). According to the earlier results, it appeared that the magnetic energy of a force-free field could exceed the energy of the corresponding open configuration. Aly (1984) has conjectured that the energy of any force-free field cannot exceed that of the corresponding open-field configuration, and suggests that the results of the Barnes-Sturrock calculations were in error. The present calculations are consistent with Aly's conjecture. When one can be confident that the calculation of the field

configuration and the field energy are not significantly affected by the external boundary, we find that the energy of the force-free field is indeed less than that of the corresponding open-field configuration. Aly in fact suggested that the earlier results could be attributed to the influence of the external boundaries in the Barnes-Sturrock calculation.

It appears, therefore, that our present calculations do not support the proposal of Barnes and Sturrock (1972) that the energy of the force-free field can exceed that of the corresponding open-field configuration. On the contrary, our present studies support the conjecture of Aly (1984) that the energy of a force-free field cannot exceed that of the corresponding open field. We intend to discuss this issue in more detail in a later article.

VI. DISCUSSION

The magneto-frictional method for computing nonlinear force-free magnetic-field configurations is simple in concept, fairly simple in operation, and appears to offer a promising approach for the calculation of general force-free magnetic-field configurations.

There are two types of problems for which one needs the capability of computing force-free fields. One of these is the calculation of the field patterns that will be produced by certain models of astrophysical systems. Our discussion of the magnetic field of a rotated sunspot is an example of such a problem. For this category of problems, the specification of the field in terms of the boundary conditions of the Clebsch variables is appropriate and convenient.

However, another class of problem is that of attempting to model the force-free field of a solar active region, in which case one attempts to find a field pattern that matches certain observational data. A vector magnetograph is now in operation at Marshall Space Flight Center (Krall et.al. 1982) that yields the vector field of an active region at the photosphere. In order to compute the corresponding force-free field, using the magneto-frictional method, it will be necessary to develop an iteration procedure. First assume a set of boundary conditions for α and β that generate the correct value of the known magnetic field and compute the corresponding force-free field. Then compare the observed transverse component of magnetic field at the photosphere with that which arises in the calculation. Based on that comparison, modify the boundary conditions for α and β. Since one is already using an iteration procedure, and proceeding in stages by adopting progressively finer grid, it is clear that one should amalgamate both iteration procedures so that gross errors in the boundary conditions are corrected at the coarse-grid level, and small-scale errors are corrected at the fine-grid level.

The calculation of real magnetic-field configurations also requires that the magneto-frictional method be executed in three dimensions rather than only two dimensions as in this article. This will no doubt require a more powerful computer than the VAX/750 that was used for the present calculations. Apart from this requirement, application of the procedure to three dimensions appears straightforward. It does seem likely that it will be more convenient to describe the field in the

form of equation (1.8) rather than in the form of equation (1.6). For instance, one could adopt any convenient specification of the variables α and β, in either the positive polarity region or the negative polarity region, and then adjust the quantity f to match the observed value of the normal magnetic field.

In addition to the extension of this procedure to three dimensions, the general approach seems appropriate for application to more general field configurations than that of a simple force-free field. For instance, in solar active regions, it often happens that one or more current sheets develop. It should be possible to divide the region occupied by the field into two regions, in each of which the magnetic field is force free, but allowing for a possible current sheet at the interface between the two regions. In principle, one should be able to apply the magneto-frictional method to each of these regions separately. However, the interface must be treated as a free boundary. If, at each point on the boundary, the magnetic field strength is different on the two sides of the boundary, then the boundary should be moved in response to the net pressure acting on the boundary at that point. By moving the boundary progressively in accordance with this rule, one should tend asymptotically to the state in which each region is force free and the interface has the appropriate equilibrium configuration. We hope to explore some of these and other developments in future articles.

A shortcoming of the present computer program is that one must calculate the field over a very large region when the field is highly stressed. This is due to the tendency of the force-free field to inflate, moving towards an open-field configuration. A possible solution to this problem is to adopt a larger mesh size in the weak-field region. Another possibility, which would be well suited for a wide range of problems, is to adopt a spherical coordinate system and arrange that the increment in radius be proportional to the radius. We shall report on the results of this approach in a later article.

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TABLE 1
MAGNETIC FIELD ENERGY AS A
FUNCTION OF ROTATION

γ_M	W
0	0.752
0.25π	0.820
0.5π	0.980
π	1.380
1.5π	1.700
2π	1.900
2.5π	2.00 (1)
openfield	2.001 (2)

(1) This estimate is only approximate due to slow convergence and proximity of the boundary. The estimate for $\gamma_M = 3\pi$ is even less certain and is not given.

(2) The energy of the open field has been estimated by adding to the energy of the field within the box an estimate of the energy of the field outside the box.

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Figure Captions

Figure 1. Contours of constant α (field lines) for the two-dimensional problem described in Section IV. (a) is the assumed initial form, and (b), (c), and (d) give the forms after 5, 15, and 60 iterations, reaspectively. These calculations were carried out on a very coarse (8×8) grid.

Figure 2. The excess energy of the computed force-free field over the energy of the exact field, as a function of iteration number, for the case desribed in Section IV.

Figure 3. Contours of constant α for the problem of cylindrical symmetry described in Section V. (a), (b), (c), (d), (e), (f), and (g) give the contours for the following values of γ_M, the relative rotation of the regions of positive and negative polarity: $0, 0.5\pi, \pi, 1.5\pi, 2\pi, 2.5\pi$, and 3π. Figure 3(h) gives the contours of constant α for the corresonding open-field configuration.

Figure 4. The energy of the cylindrical force-free field model of Section V as a function of γ_M, which measures the relative rotation of the positive polarity region and the negative polarity region. Also shown is the energy of the corresponding open-field configuration.

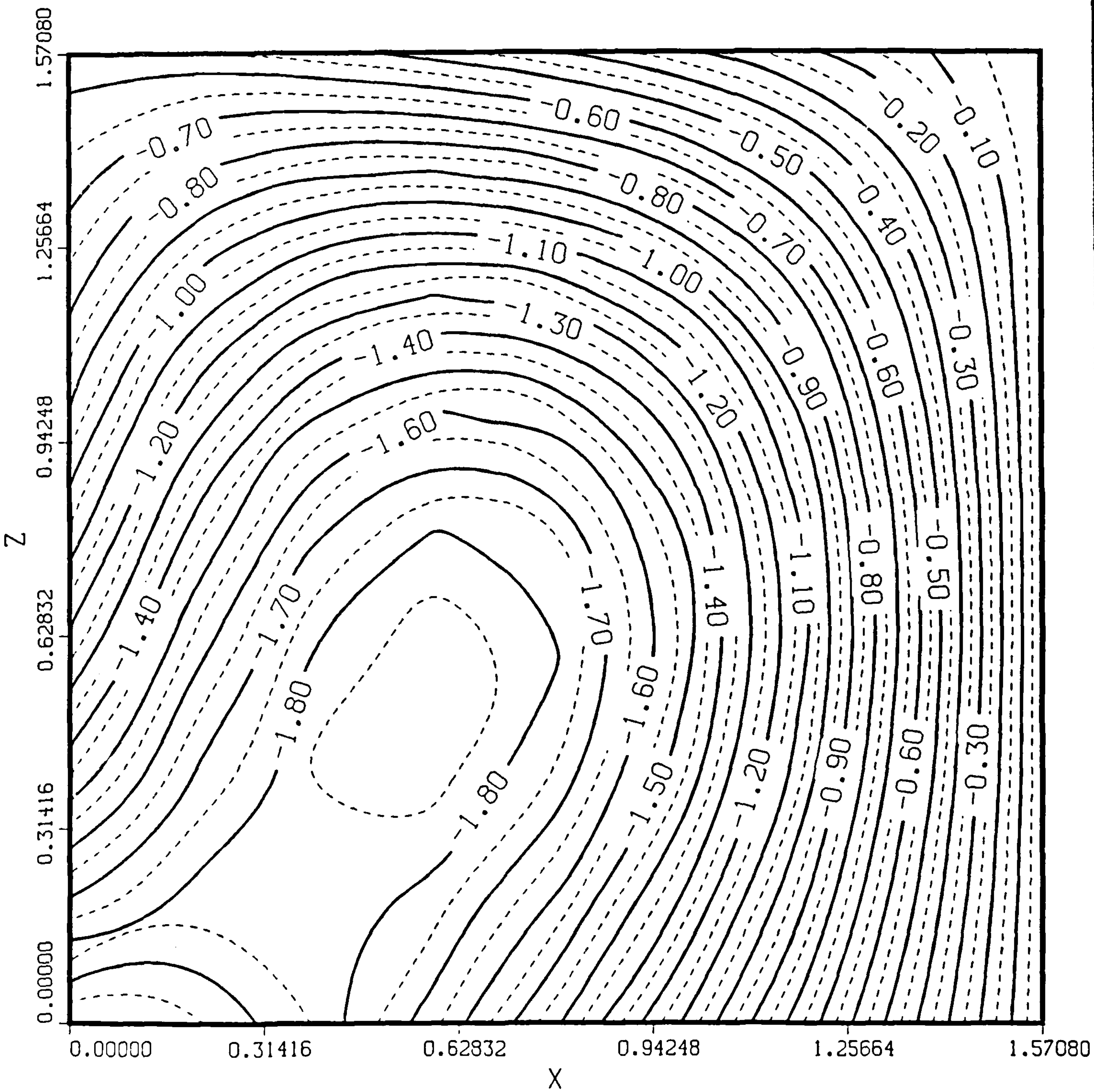
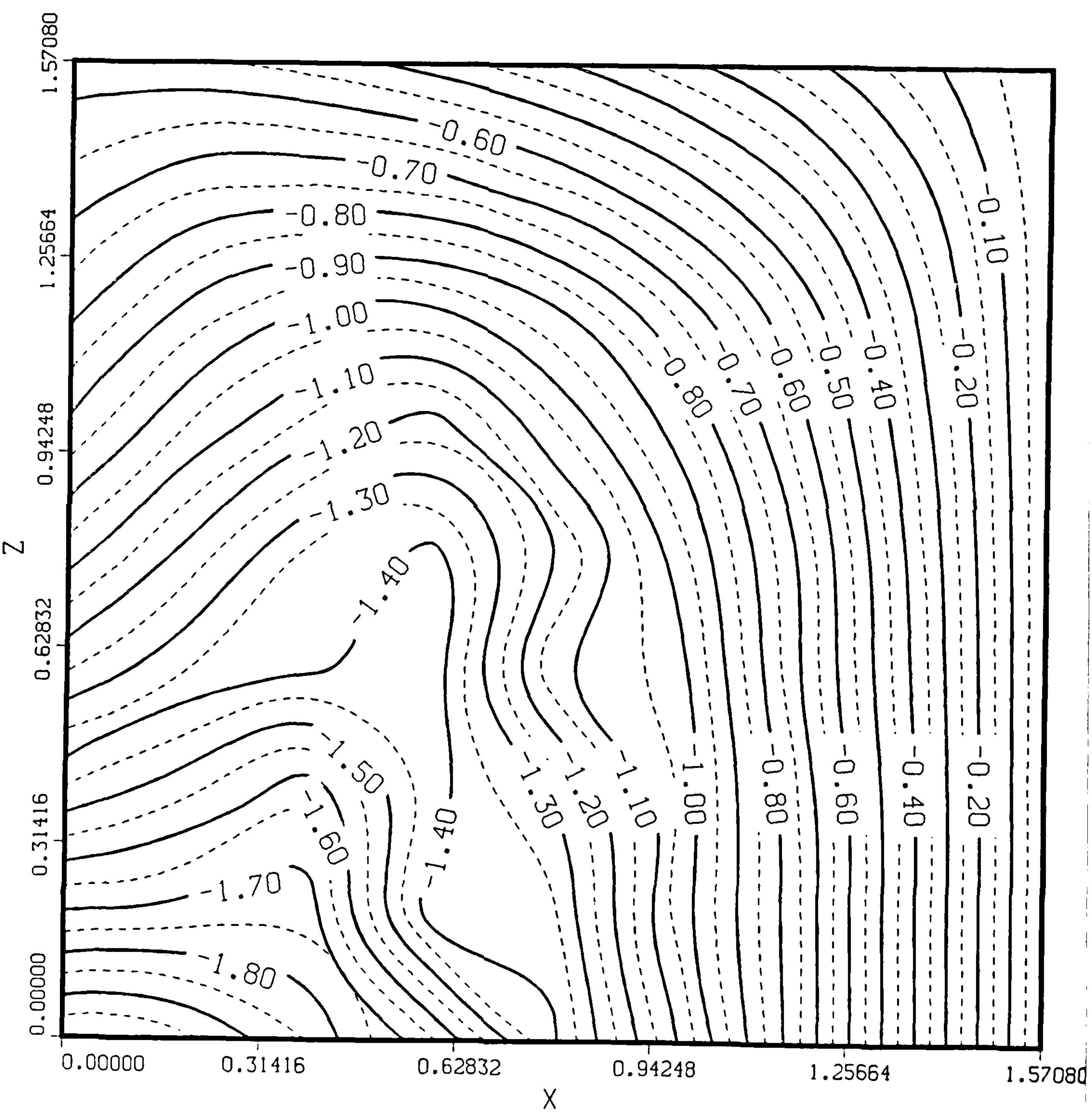
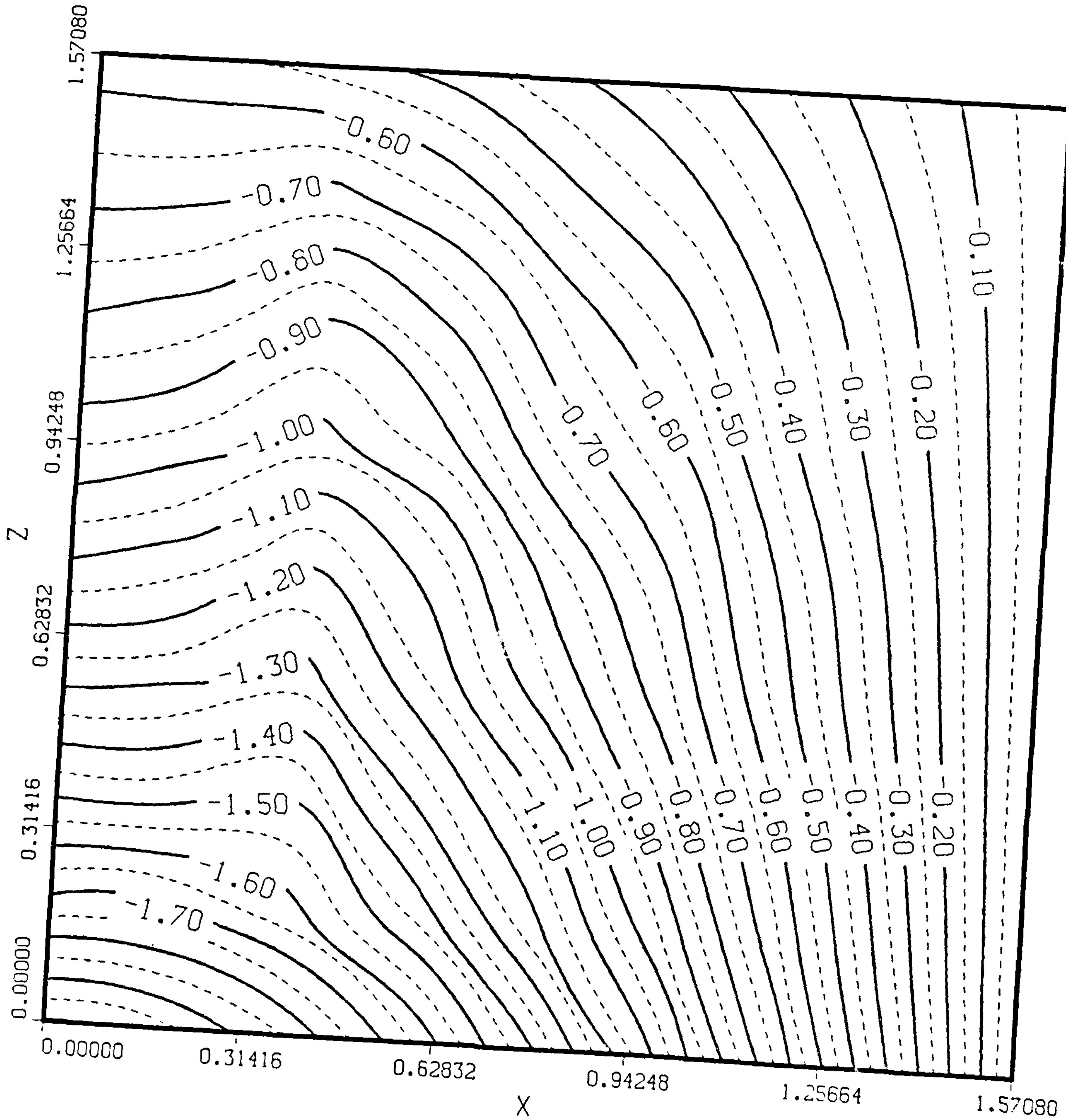

Z
X
0.00000
0.31416
0.62832
0.94248
1.25664
1.57080
-0.60
-0.50
-0.20
-0.10
-0.70
-0.80
-0.80
-0.70
-0.40
-1.10
-1.00
-1.00
-1.30
-0.60
-0.30
-1.40
-0.90
-1.20
-1.20
-1.60
-1.40
-1.70
-1.70
-1.40
-1.10
-0.80
-0.50
-1.80
-1.60
-1.80
-1.50
-1.20
-0.90
-0.60
-0.30

FIGURE 1a



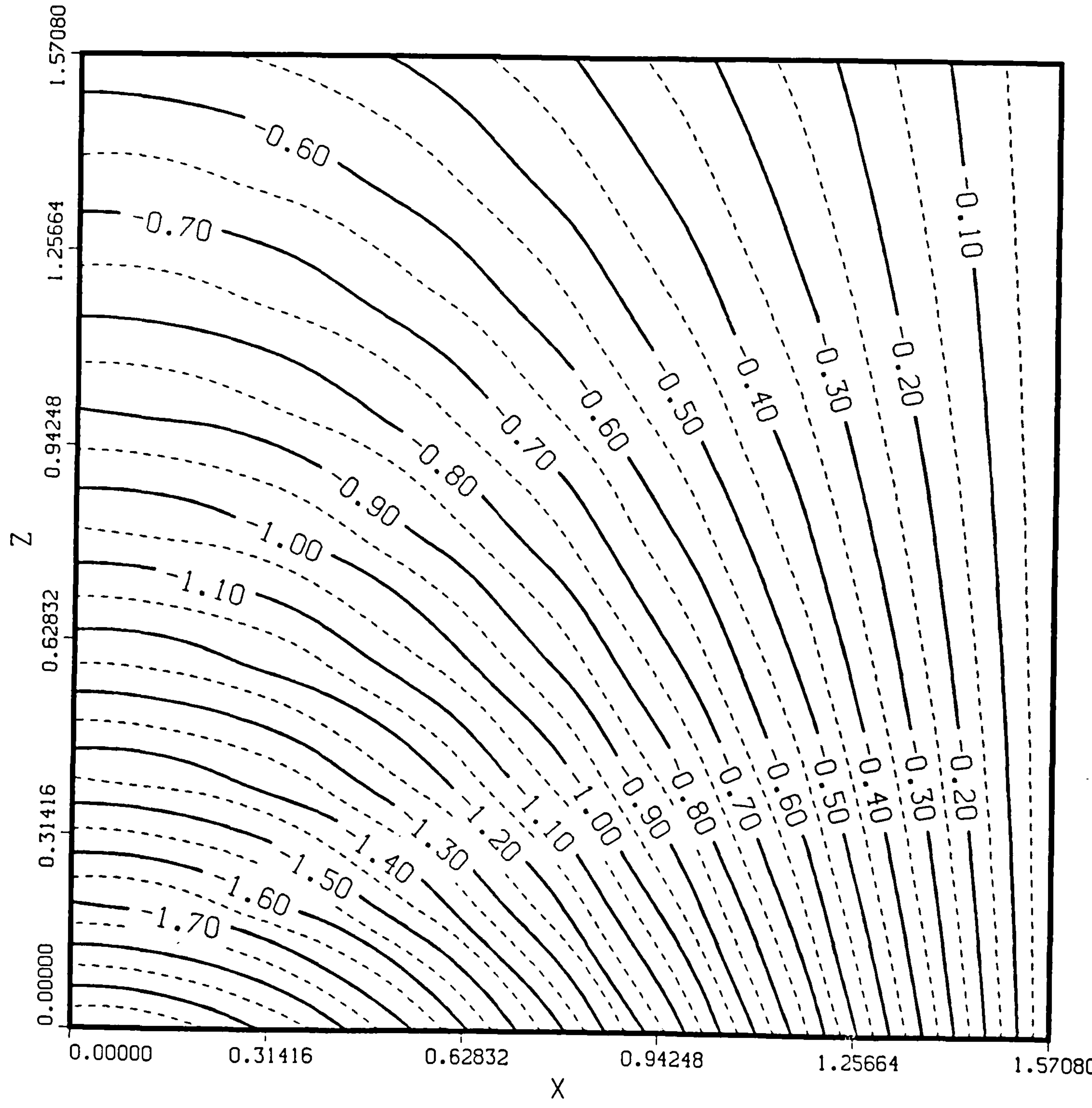
1.57080
1.25664
0.94248
0.62832
0.31416
0.00000
Z
0.00000
0.31416
0.62832
0.94248
1.25664
1.57080
X
-0.60
-0.70
-0.80
-0.90
-1.00
-1.10
-1.20
-1.30
-1.40
-1.50
-1.60
-1.70
-1.80
-1.40
-1.30
-1.20
-1.10
-1.00
-0.80
-0.60
-0.40
-0.20
-0.80
-0.70
-0.60
-0.50
-0.40
-0.20
-0.10

FIGURE 1b



1.57080
1.25664
0.94248
0.62832
0.31416
0.00000
Z
0.00000
0.31416
0.62832
0.94248
1.25664
1.57080
X
-0.60
-0.70
-0.80
-0.90
-1.00
-1.10
-1.20
-1.30
-1.40
-1.50
-1.60
-1.70
-0.10
-0.70
-0.60
-0.50
-0.40
-0.30
-0.20
-1.10
-1.00
-0.90
-0.80
-0.70
-0.60
-0.50
-0.40
-0.30
-0.20

FIGURE 1c



-0.60
-0.70
-0.80
-0.90
-1.00
-1.10
-1.20
-1.30
-1.40
-1.50
-1.60
-1.70
-0.50
-0.40
-0.30
-0.20
-0.10
Z
X
0.00000
0.31416
0.62832
0.94248
1.25664
1.57080

FIGURE 1d

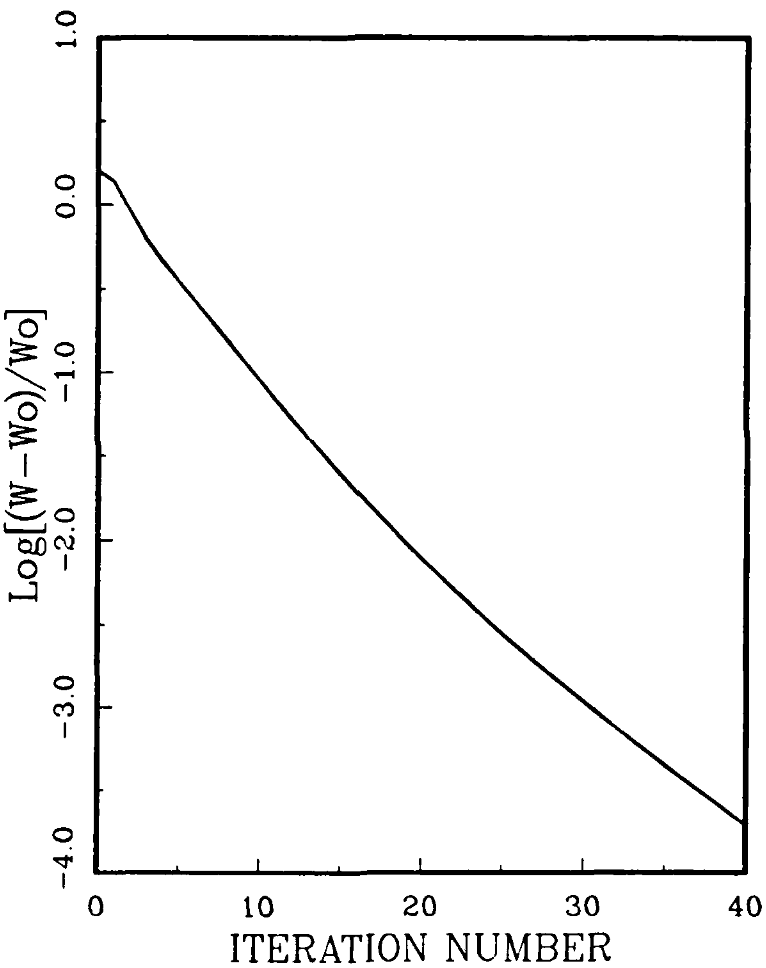
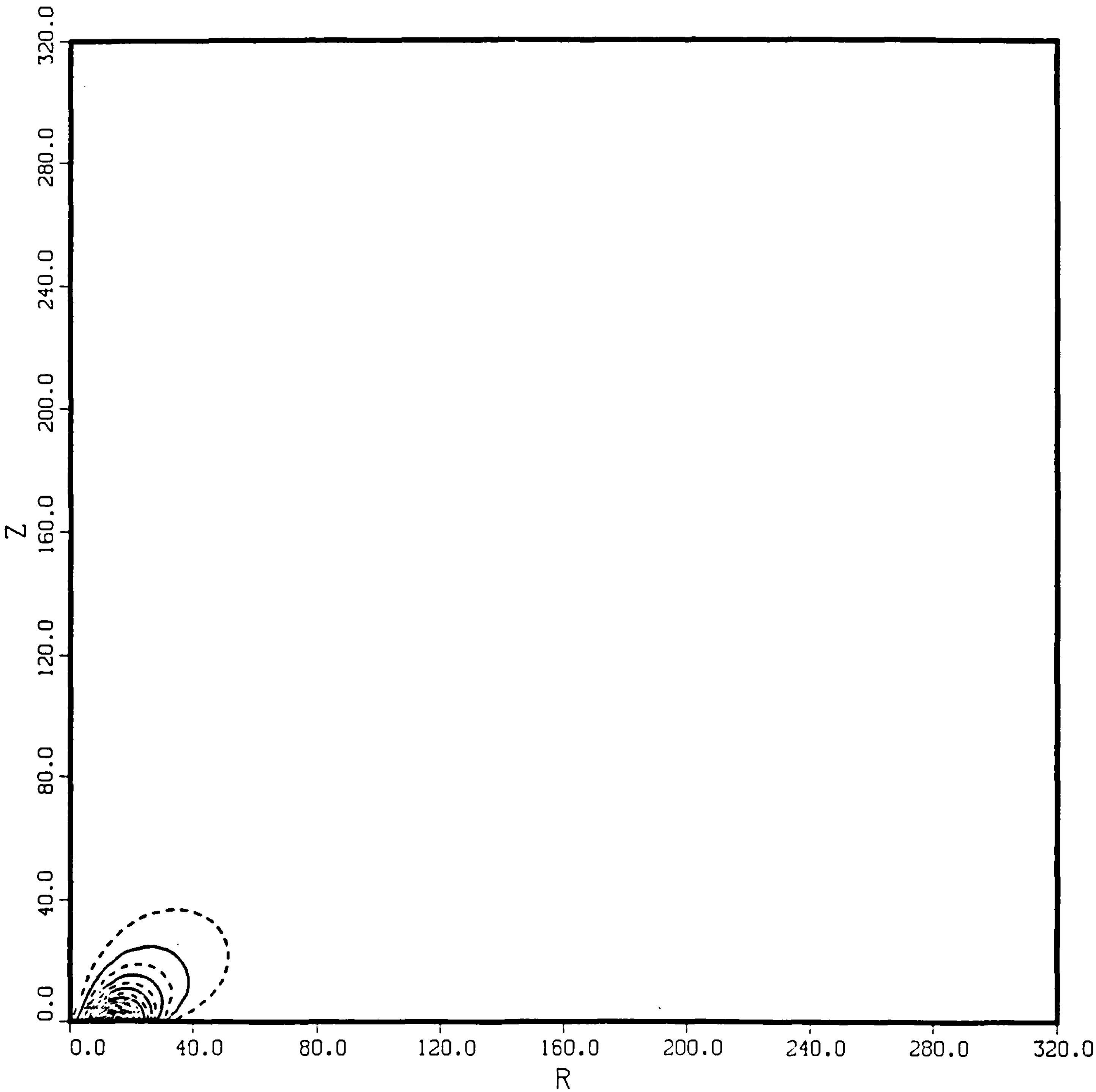
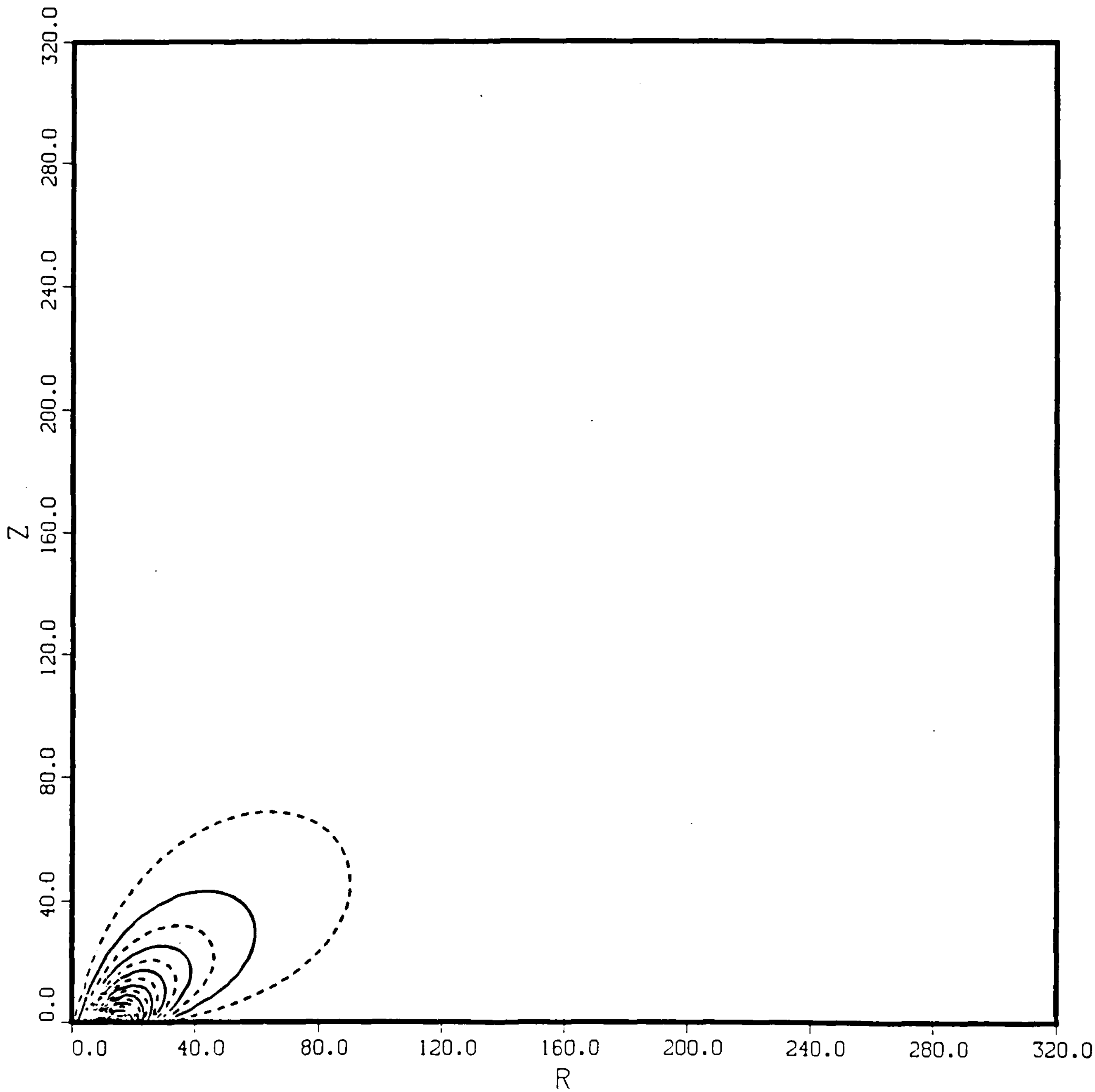


FIGURE 2



320.0
280.0
240.0
200.0
160.0
120.0
80.0
40.0
0.0
Z
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R

FIGURE 3a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240.0
200.0
160.0
120.0
80.0
40.0
0.0
Z
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R

FIGURE 3b

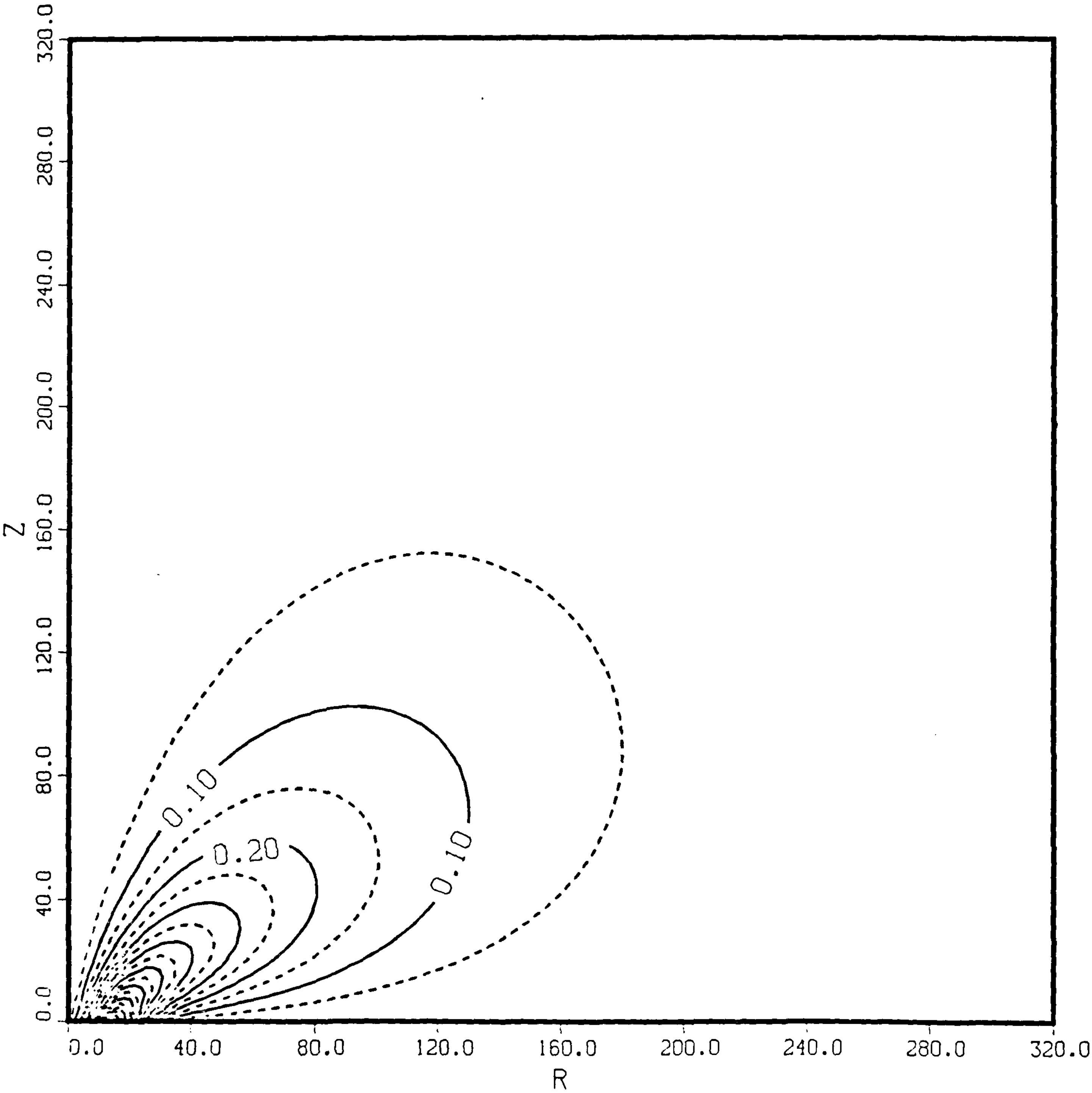
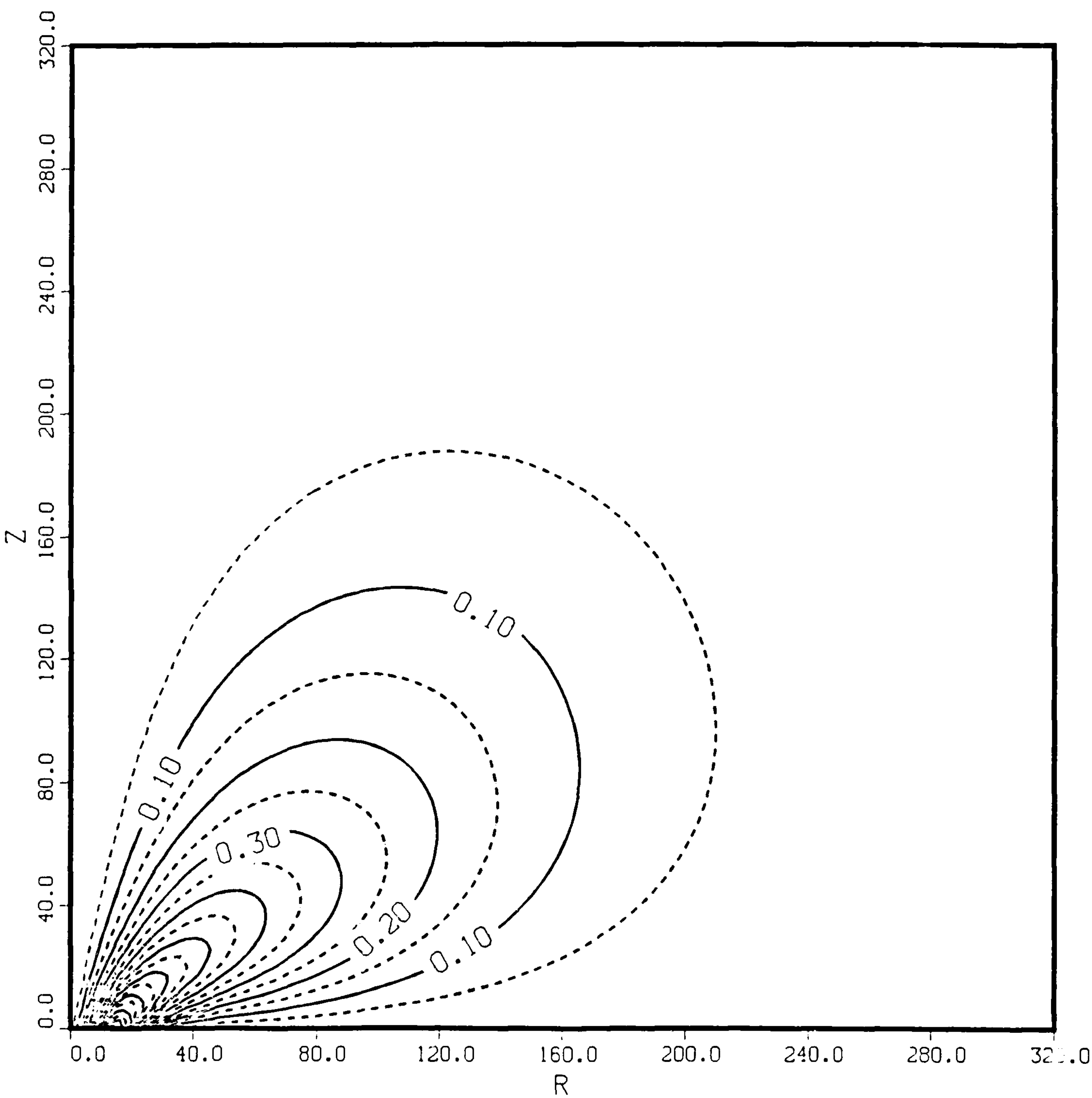
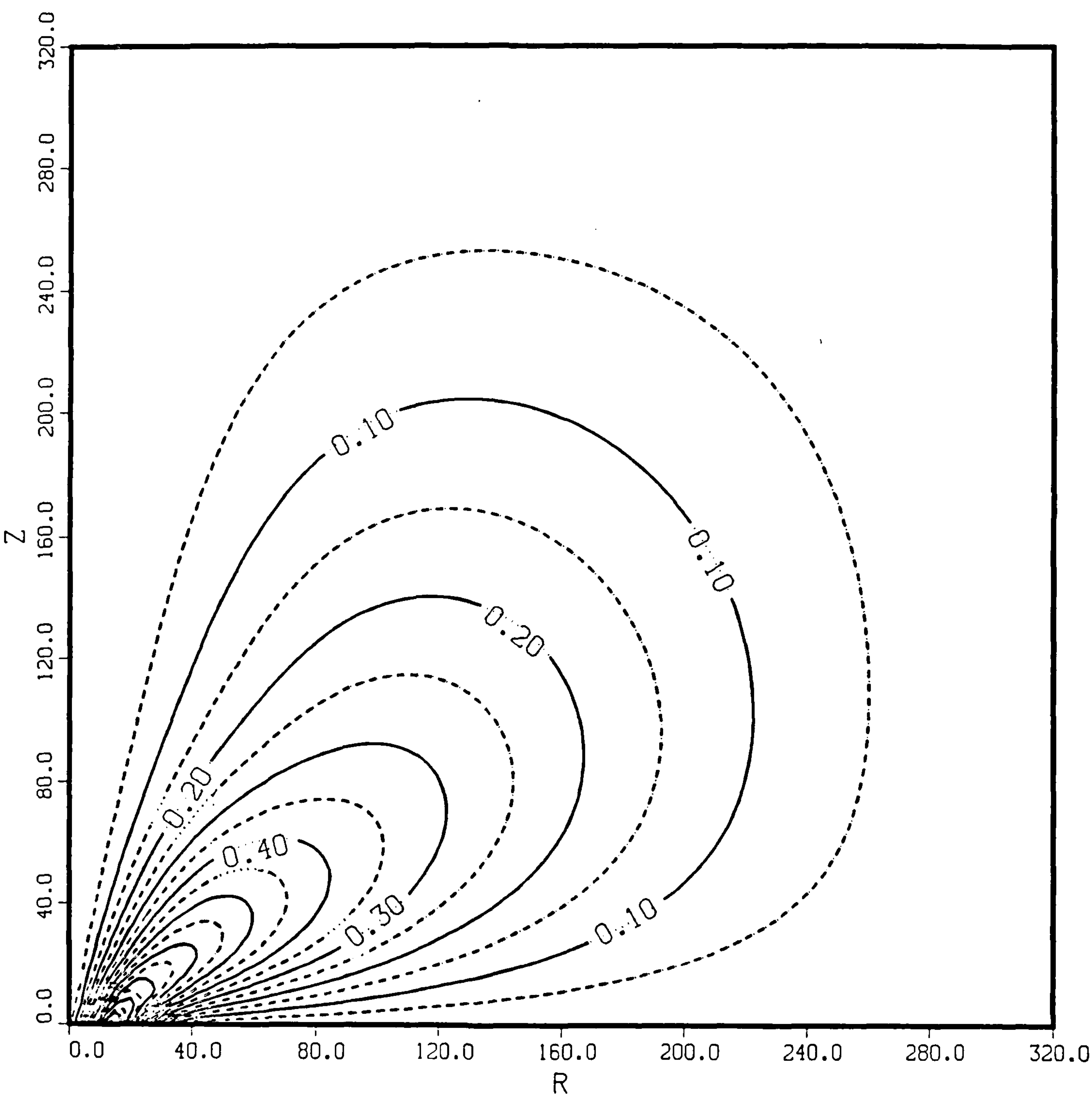

320.0
280.0
240.0
200.0
160.0
Z
120.0
80.0
40.0
0.0
0.10
0.20
0.10
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R

FIGURE 3c



320.0
280.0
240.0
200.0
160.0
120.0
80.0
40.0
0.0
Z
0.10
0.10
0.30
0.20
0.10
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
R

FIGURE 3d



320.0
280.0
240.0
200.0
160.0
120.0
80.0
40.0
0.0
Z
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R
0.10
0.10
0.10
0.20
0.20
0.40
0.30
0.10

FIGURE 3e

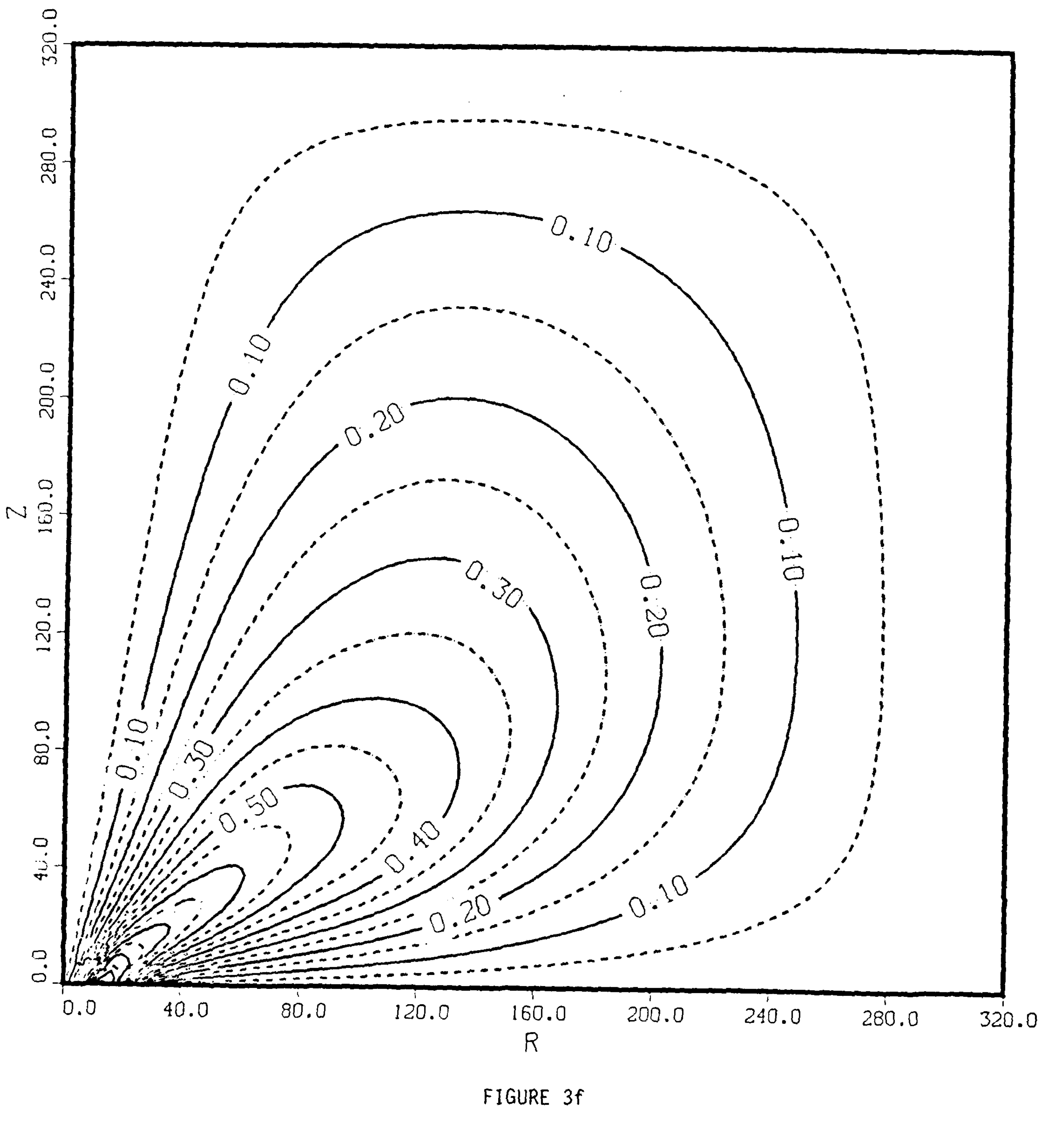
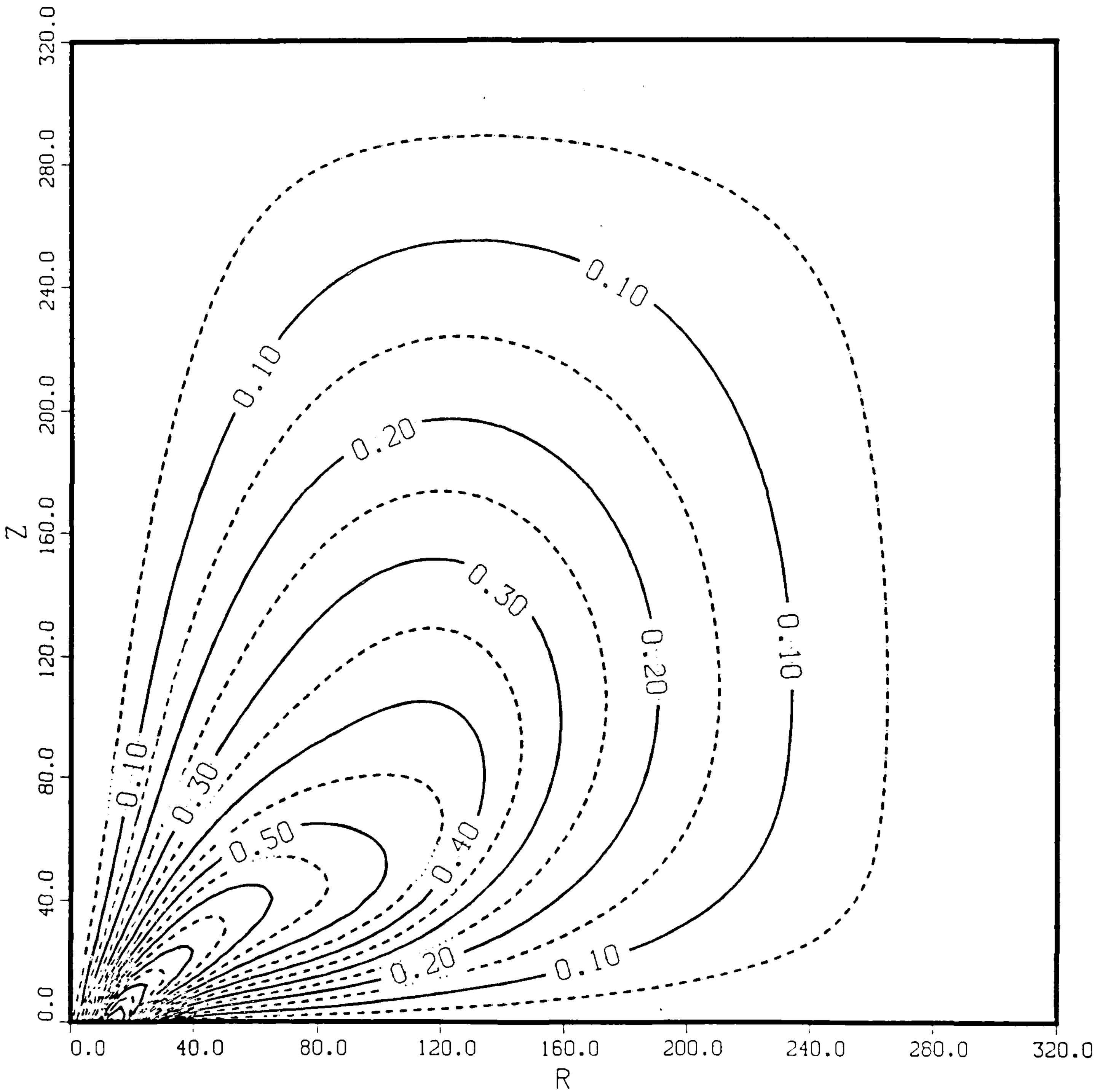
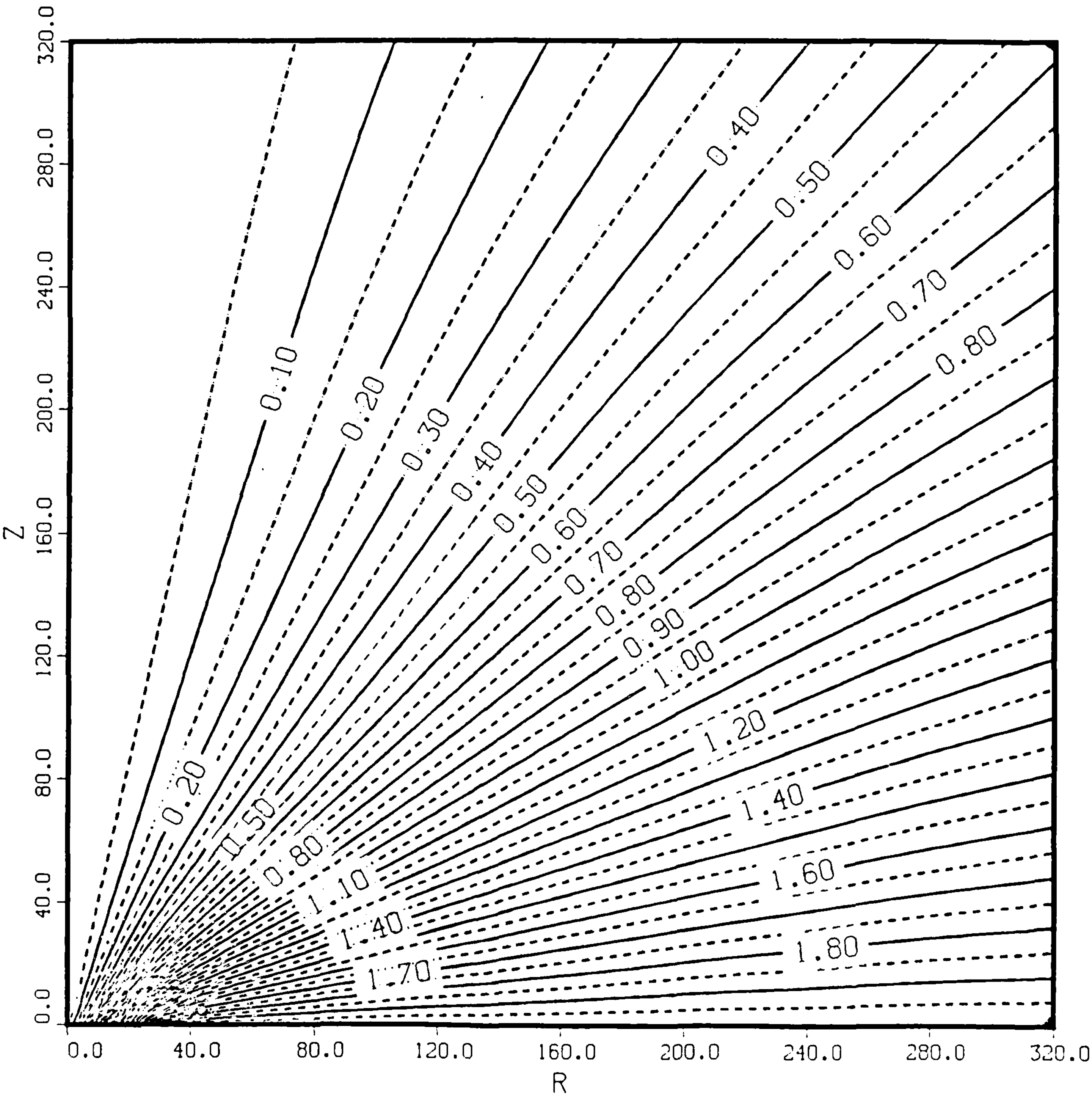


FIGURE 3f



320.0
280.0
240.0
200.0
160.0
120.0
80.0
40.0
0.0
Z
0.0
40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R
0.10
0.10
0.20
0.30
0.20
0.10
0.10
0.10
0.30
0.50
0.40
0.20
0.10

FIGURE 3g



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200.0
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120.0
80.0
40.0
0.0
Z
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40.0
80.0
120.0
160.0
200.0
240.0
280.0
320.0
R
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
1.20
1.40
1.60
1.80
1.10
1.70

FIGURE 3h

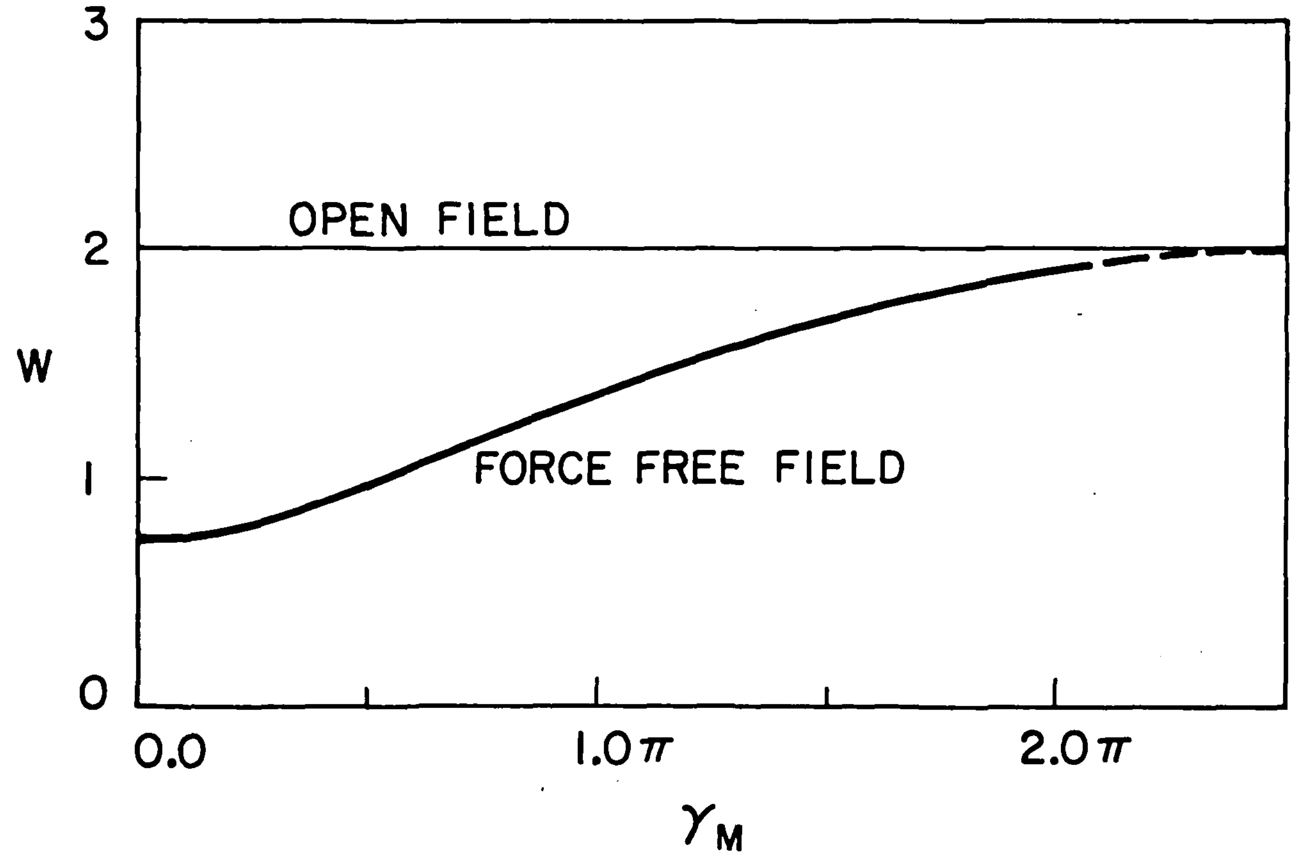

FIGURE 4

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